Kan and Grothendieck Commute Together

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Abstract

This early idea abstract starts with a motivating question: how can a free category be computed on a decomposition of graphs? As free categories may represent either languages on abstract machines (in the internal case) or a class of computational problems called the algebraic path problem (in the enriched case) the answer to our motivating question promises to give insight into the compositionality of a wide class of problems. We answer our question through generalization: there is a square of functors expressing the commutation of generalized Grothendieck constructions and left Kan extensions which specializes to the case of free categories on decompositions of graphs.

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0.1 Decompositions and Generalizing Grothendieck

The "category theorist's graph" i.e. a pair of functions $s, t : E \to V$ may be known to automata theorists as a labelled transition system. These transition systems are often modular; for example, in Figure 1, there is a Petri net from the 2021 Model Checking Contest which may be unfolded into a labelled transition system G representing its behavior [4]. In this case, we seek general categorical methods to reason about the behavior of such systems by leveraging their inherent modularity.

We take a deceptively simple approach to graph decompositions: they are *just* graph morphisms. The idea is that for each $y \in Y$, the preimage $g^{-1}(y)$ is the "bag" over y and for each $x \in X$, the preimage $f^{-1}(x)$ is the "adhesion" over the edge x. In the case of Figure 1, the bags are the clients or servers and the adhesions are the edges connecting them. Graph decompositions may be made complicated again, as shown below:

$$\begin{array}{ccc} E & \overset{s}{\underset{t}{\rightrightarrows}} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \overset{u}{\underset{v}{\rightrightarrows}} & Y \end{array} \quad \cong \quad \begin{array}{ccc} X & \overset{u}{\underset{v}{\rightrightarrows}} & Y \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathsf{Spans} & \overset{src}{\underset{tgt}{\rightrightarrows}} & \mathsf{Sets} \end{array}$$

This equivalence is reminiscient of the Grothendieck construction; the forward direction turns a graph morphism into its inverse image and the backwards direction takes dependent pairs. Indeed, it is a Grothendieck construction obtained by internalizing a category T into the category ISet of indexed sets i.e. large functions $X \to \mathsf{Sets}$. A functor $T \to \mathsf{ISet}$ is interpreted as a decomposition of models of T. There is always a Grothendieck construction for such functors:

▶ **Theorem 1.** *For every small category T, there is an equivalence of categories*

$$\textstyle\int_T : [T, \mathsf{ISet}] \cong [T, \mathsf{Set}^{\to}]$$

To see the graph Grothendieck construction as a special case, we must construct equivalence of categories $[\mathsf{ThGr}, \mathsf{ISet}] \cong \mathsf{Graph}/\mathbb{S}\mathrm{pan}_{Gr}$ where ThGr is the "walking graph".

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0.2 Free Categories and Compositionality Squares

The free category on a transition system, such as the one G derived from Figure 1, has morphisms as all possible sequences of communications that may occur in the network (in other words $F(G)(x,y)$ is the language of G which starts in x and ends in y). Because G is actually a decomposition of graphs, $D : S \to \mathbb{S}\mathrm{pan}_{Gr}$, we can find its language in two ways. The first way $F_{\to}(\int_{\mathsf{ThGr}}(G))$ glues the decomposition together and then takes its free category. The second way, does this in the reverse order $\int_{\mathsf{ThCat}}(F_{ind}((G)))$; first the free category is found as a graph decomposition and then it is glued together into a category. These two ways of finding the transition system's language are represented as the two paths in the square below:

$$\begin{array}{ccc} \mathsf{Graph}/\mathbb{S}\mathrm{pan}_{Gr} & \xrightarrow{\int_{\mathsf{ThGr}}} & \mathsf{Graph}^{\to} \\ \downarrow F_{ind} & \cong & F_{\to}\downarrow \\ \mathsf{Cat}/\mathbb{S}\mathrm{pan}_{Cat} & \xrightarrow[\int_{\mathsf{ThCat}}]{} & \mathsf{Cat}^{\to} \end{array}$$

The first way, $F_{\to}(\int_{\mathsf{ThGr}}(G))$ does not leverage the compositional structure and just finds the language all at once. The second way, $\int_{\mathsf{ThCat}}(F_{ind}((G)))$ finds the language more cleverly, by first finding the language of the shape graph S and then casting the results on the bags and adhesions. The square commuting up to natural isomorphism proves the correctness of the clever way because it means that it agrees with the naive approach. It is an open question whether the bottom path of this square may inform an algorithm for finding the language of the graph decomposition in a compositional way. Because taking free categories is a left Kan extension along a morphism of theories $\mathsf{ThGr} \to \mathsf{ThCat}$, we may see the above square as a special case of the one below.

$$\begin{array}{ccc} [T, \mathsf{ISet}] & \xrightarrow{\int_{T}} & [T, \mathsf{Set}^{\to}] \\ \mathsf{Lan}_f(-)\downarrow & \cong & \downarrow\mathsf{Lan}_f(-) \\ [T', \mathsf{ISet}] & \xrightarrow[\int_{T'}]{} & [T', \mathsf{Set}^{\to}] \end{array}$$

This suggests that more generally, the free model of an algebraic theory generated by another, has a "compositional algorithm" given by taking the clever path of the square. In particular, there is a theory whose models are Petri nets and where the left Kan extension turns a Petri net into the operational semantics given in [3] making Petri nets another potential application for this framework. Another open question is how to generalize the square for free categories to enriched categories. As argued in [1], [2], the answer to this question may give compositional techniques for finding solutions to algebraic path problems.

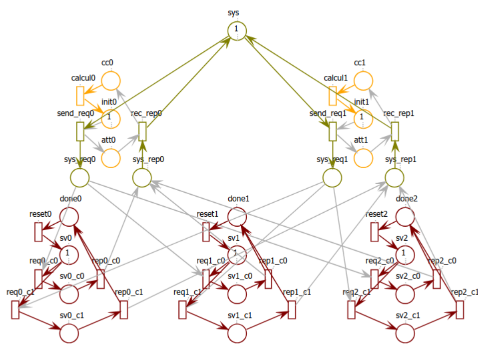

Figure 1 A Petri net represent the interaction of clients and servers from the 2021 MCC

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2 Jade Master. How to compose shortest paths. In *Structure Meets Power Workshop (Contributed Talks)*, page 15, 2022.

3 José Meseguer and Ugo Montanari. Petri nets are monoids. *Information and Computation*, 88(2):105–155, 1990.

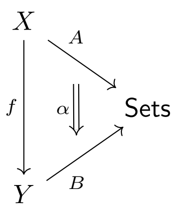
4 Fabrice Kordon Tra My Nguyen. Model: Serversandclients. Model Checking Contest 2021. Available at https://mcc.lip6.fr/2025/pdf/ServersAndClients-form.pdf.

A Omitted Definitions and Proof Sketches

▶ **Definition 2.** *Let* Sets *be the large set of all small sets* $X, Y, Z, \dots$ *and let* Spans *be the large set of all pairs of functions* $X \xleftarrow{a} A \xrightarrow{b} Y$. *Let src and tgt be the large functions sending each span to its source and target sets respectively. The there is a graph* $\mathbb{S}\mathrm{pan}_{Gr}$ *defined by*

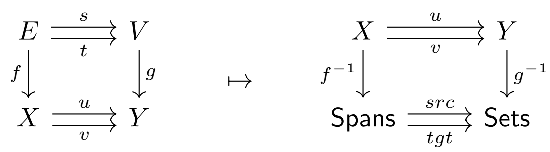
$$\mathsf{Spans} \underset{tgt}{\overset{src}{\rightrightarrows}} \mathsf{Sets}$$

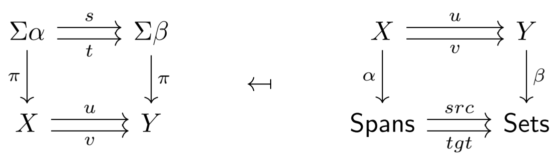
▶ **Definition 3.** *An indexed set is a large function* $X \to \mathsf{Sets}$ *and a morphism of indexed sets is a function* f *between the base sets along with an indexed* $\alpha_x : A(x) \to B(f(x))$ *as shown below.*



This defines a category ISet *of indexed sets and there morphisms.*

▶ **Proposition 4.** *There is an equivalence of categories* $\mathsf{Graph}^{\to} \cong \mathsf{Graph}/\mathbb{S}\mathrm{pan}_{Gr}$ *given by*





where Σ *sends an indexed set to its set of dependent pairs and* π *is the first projection of these pairs.*

The above theorem is a special case of the next one.

▶ **Theorem 5.** *For every small category* T, *there is an equivalence of categories*

$$\textstyle\int_T : [T, \mathsf{ISet}] \cong [T, \mathsf{Set}^{\to}]$$

Proof. To construct this equivalence, first define an equivalence $\mathsf{ISet} \cong \mathsf{Set}^{\rightarrow}$. Going left to right, we send an indexed set $X \xrightarrow{A} \mathsf{Sets}$ to the dependent pair and projection $\Sigma A \xrightarrow{\pi} X$. Going right to left we send a function to its preimage mapping. Applying the 2-functor $[T, -] : \mathsf{Cat} \to \mathsf{Cat}$ preserves this equivalence because every 2-functor preserves equivalences. ◀

▶ **Proposition 6.** *There is an equivalence*

$$\mathsf{Graph}/\mathbb{S}\mathrm{pan}_{Gr} \cong [\mathsf{ThGr}, \mathsf{ISet}]$$

Proof. A functor $\mathsf{ThGr} \to \mathsf{ISet}$ has two indexed sets $e : E \to \mathsf{Sets}$ and $v : V \to \mathsf{Sets}$ along with indexed morphisms $s : e \Rightarrow v$ and $t : e \Rightarrow v$. E, V and the base components of s and t assemble into an ordinary graph G which will be the domain of a graph morphism $G \to \mathbb{S}\mathrm{pan}_{Gr}$. The actual mappings of this graph morphism are given by e and v where the extra spans necessary in the edge component are coming from the indexed portions of s and t. ◀

▶ **Proposition 7.** *There is an equivalence*

$$\mathsf{Cat}/\mathbb{S}\mathrm{pan}_{Cat} \cong [\mathsf{ThCat}, \mathsf{ISet}]$$

Proof. This equivalence will be very similar to the previous one, as the theory of categories is just the theory of graphs with additional axioms for composition and identities satisfying the usual axioms. However there is some subtlety as $\mathbb{S}\mathrm{pan}_{Cat}$ is best thought of as a double category and the functors into it will be lax since composition of spans by pullback is only defined up to isomorphism and not every element of the pullback will specified by a functor $\mathsf{ThCat} \to \mathsf{Set}$. ◀

▶ **Proposition 8.** *There is an adjunction*

$$\mathsf{Graph}/\mathbb{S}\mathrm{pan}_{Gr} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Cat}/\mathbb{S}\mathrm{pan}$$

Proof. It is well known that there is an adjunction $F : \mathsf{Graph} \leftrightarrow \mathsf{Cat} : U$ whose right adjoint is forgetful and whose left adjoint sends a graph G to the category whose morphisms are given by $\sum_{n \geq 0} G^n$ where $\sum$ indicates coproduct and G^n indicates the n-fold pullback of the graph with itself. This adjunction provides an equivalence between graph morphisms $G \to U(C)$ and functors $F(G) \to C$. We may use this adjunction to form a left adjoint F_{ind} on $\mathsf{Graph}/\mathbb{S}\mathrm{pan}_{Gr}$ by sending a morphism $G \to \mathbb{S}\mathrm{pan}_{Gr}$ to its "mate" $FG \to \mathbb{S}\mathrm{pan}_{Cat}$ with the understanding that $\mathbb{S}\mathrm{pan}_{Gr} \cong U(\mathbb{S}\mathrm{pan}_{Cat})$. ◀

▶ **Proposition 9.** *There is an adjunction*

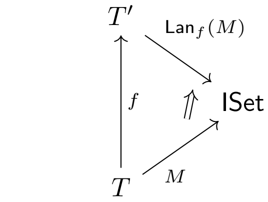
$$\mathsf{Graph}^{\rightarrow} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Cat}^{\rightarrow}$$

Proof. The left adjoint $F_{\rightarrow} : \mathsf{Graph}^{\rightarrow} \to \mathsf{Cat}^{\rightarrow}$ is defined by sending a morphism $H \xrightarrow{f} G$ to the functor $F(H) \xrightarrow{F(f)} F(G)$. ◀

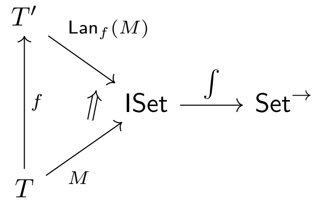
▶ **Theorem 10.** *The square of functors below commutes up to natural isomorphism.*

$$\begin{array}{ccc} [T, \mathsf{ISet}] & \xrightarrow{\int_T} & [T, \mathsf{Set}^{\rightarrow}] \\ {\scriptstyle \mathsf{Lan}_f(-)}\downarrow & \cong & \downarrow{\scriptstyle \mathsf{Lan}_f(-)} \\ [T', \mathsf{ISet}] & \xrightarrow[\int_{T'}]{} & [T', \mathsf{Set}^{\rightarrow}] \end{array}$$

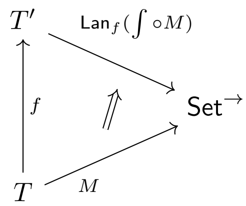
Proof. Because ISet has all small colimits, we may always form the left Kan extension



when T and T' are small categories. Then we may compose it with the equivalence with the above equivalence to get



alternatively, we may take the Kan extension



we wish to know if there is a natural isomorphism $\mathsf{Lan}_f(\int \circ M) \cong \int \circ \mathsf{Lan}_f M$. Indeed there always will be an isomorphism because Kan extensions are preserved by left adjoints and in particular equivalences. ◀